# ADDITIVE REPRESENTATION IN THIN SEQUENCES, IV: LOWER BOUND METHODS

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Abstract. We describe a method for establishing that values from a fixed polynomial sequence are represented frequently by some prescribed sum of powers of natural numbers. As an illustration of this method, we show that for at least $X^{129/136}$ of the integers $n$ with $1 \le n \le X$, a fixed quadratic polynomial $\phi(n)$ may be written as the sum of five cubes of positive integers. A similar result is established for the sum of a square and three cubes of positive integers.

**1. Introduction.** Technology currently available in the additive theory of numbers frequently lacks sufficient power to establish that all large natural numbers (perhaps constrained by inherent congruence conditions) are represented in a prescribed form. Even the weaker conclusion that almost all natural numbers are thus represented is often beyond our grasp. In such circumstances, one must seek instead to establish that natural numbers are *frequently* represented in the prescribed manner. In previous parts of this series (see Brüdern, Kawada and Wooley [4, 5, 6]), we have developed and explored an approach to additive problems in which one seeks to show that almost all natural numbers in a fixed polynomial sequence are represented in some prescribed manner, thereby establishing non-trivial estimates for exceptional sets in thin sequences. We now adapt this technology so as to handle analogous problems in which a non-trivial estimate for the exceptional set remains inaccessible, seeking instead to demonstrate that integers from the fixed polynomial sequence are frequently represented. As such, our results provide information which may be regarded as more enlightening than that available hitherto in those problems accessible to our methods.

Rather than embark immediately on a discussion of the general features of our rather flexible method, we instead follow the pattern established in previous parts of this series, illustrating the type of conclusion now available with a discussion of Waring's problem for cubes. So far as this paper is concerned, the motivating conjecture associated with the latter topic is the assertion that a positive proportion of the natural numbers are represented as the sum of three cubes of positive integers. The sharpest conclusion currently available in this direction shows that when $x$ is large, slightly more than $x^{11/12}$ of the natural numbers up to $x$ are thus represented (see Vaughan [14] for a slightly weaker conclusion, and Wooley [19] for the most recent progress on this problem). While it is known that almost all natural numbers are the sum of four positive cubes (see Brüdern [3], and Wooley [19] for the latest developments), rather less is known if one restricts the aforementioned natural numbers to a thin sequence of polynomial values. In a previous part of this series (see Brüdern, Kawada and Wooley [4]), we established that for any fixed quadratic polynomial $\phi \in \mathbb{Z}[t]$, all but $O(x^{19/28})$ of the natural numbers $n$ up to $x$ satisfy the property that $\phi(n)$ is represented as the sum of six cubes of positive integers. A similar, though somewhat weaker, conclusion was also provided for cubic polynomials $\phi$. We presently lack sufficient power to establish that almost all values of a fixed quadratic polynomial are the sum of five cubes of positive integers. A variant of the method introduced in [4], however, shows that the former values occur frequently as sums of five cubes.

It is convenient henceforth to describe a polynomial $\phi \in \mathbb{Q}[t]$ as being an *integral polynomial* if, whenever the parameter $t$ is an integer, then the value $\phi(t)$ is also an integer. When $\phi$ is an integral polynomial, denote by $\mathcal{N}_\phi(X)$ the number of integers $n$, with $1 \le n \le X$, for which $\phi(n)$ is the sum of

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five cubes of positive integers. Following work on some auxiliary estimates in §§2 and 3, we advance in §4 to establish the lower bound for $\mathcal{N}_\phi(X)$ recorded in the following theorem.

**Theorem 1.1.** *Let $\phi$ be an integral quadratic polynomial with positive leading coefficient. Then $\mathcal{N}_\phi(X) \gg_\phi X^{129/136}$.*

We discuss a second consequence of our methods in §5, where we establish that integers represented as the sum of a square and three cubes of positive integers occur frequently amongst the values of certain quadratic polynomial sequences. We consider an integral quadratic polynomial of the shape $\phi(t) = t^2 - a$, for a fixed integer $a$, or $\phi(t) = \Xi - t^2$, for a sufficiently large natural number $\Xi$. In the latter respective cases, when $N$ is sufficiently large in terms of $a$, or when $N = \Xi^{1/2}$, we denote by $\mathcal{X}_\phi(N)$ the number of integers $n$ with $1 \le n \le N$ for which $\phi(n)$ is the sum of a square and three cubes.

**Theorem 1.2.** *Let $\phi$ be a quadratic polynomial of the shape discussed above. Then for any fixed positive number $\varepsilon$, one has $\mathcal{X}_\phi(N) \gg N^{20/21-\varepsilon}$.*

We remark that subject to a suitable extension of the work of Hooley [12] and Daniel [9, 10], one can replace the quadratic polynomials $\phi$ discussed above by arbitrary quadratic polynomials.

It may be illuminating to describe our general strategy for lower bound problems. Given a positive integer $n$ and sets $\mathcal{A}_1, \dots, \mathcal{A}_s$ of positive integers, we investigate the number $r(n)$ of representations of $n$ in the shape

$$n = a_1 + a_2 + \dots + a_s, \tag{1.1}$$

with $a_i \in \mathcal{A}_i$ $(1 \le i \le s)$. When $\mathcal{B} \subseteq \mathbb{N}$, we define $N_\mathcal{B}(X)$ to be the number of integers $n \in [1, X] \cap \mathcal{B}$ possessing a representation in the shape (1.1). In order that it be possible to establish a non-trivial lower bound for $N_\mathcal{B}(X)$, the equation (1.1) must plainly possess infinitely many solutions with $n \in \mathcal{B}$ and $a_i \in \mathcal{A}_i$ $(1 \le i \le s)$. This important prerequisite excludes, for the present, the possibility of establishing an analogue of Theorem 1.1 in which only four cubes are employed.

Define the generating functions

$$f_i(\alpha) = \sum_{x \in \mathcal{A}_i \cap [1,X]} e(\alpha x) \quad (1 \le i \le s),$$

where $e(z)$ denotes $e^{2\pi i z}$. Then by orthogonality, one has

$$r(n) = \int_0^1 f_1(\alpha) \dots f_s(\alpha) e(-\alpha n) d\alpha.$$

Denote by $\mathcal{Z}(X)$ the set of integers $n$ with $n \in \mathcal{B} \cap [1, X]$ which possess a representation in the shape (1.1). Define also the generating functions

$$k(\alpha) = \sum_{n \in \mathcal{B} \cap [1,X]} e(\alpha n) \quad \text{and} \quad K(\alpha) = \sum_{n \in \mathcal{Z}(X)} e(\alpha n).$$

Then plainly,

$$\sum_{n \in \mathcal{B} \cap [1,X]} \int_0^1 f_1(\alpha) \dots f_s(\alpha) e(-\alpha n) d\alpha = \sum_{n \in \mathcal{Z}(X)} \int_0^1 f_1(\alpha) \dots f_s(\alpha) e(-\alpha n) d\alpha,$$

whence

$$\int_0^1 f_1(\alpha) \dots f_s(\alpha) k(-\alpha) d\alpha = \int_0^1 f_1(\alpha) \dots f_s(\alpha) K(-\alpha) d\alpha. \tag{1.2}$$

Our strategy is now to obtain a lower bound for the left hand side of (1.2) by means of the Hardy-Littlewood method, or any viable substitute, and also to obtain an upper bound for the right hand side of (1.2), depending on $\text{card}(\mathcal{Z}(X))$, by appealing to Hölder's inequality. By combining these bounds, we deduce a lower bound for $\text{card}(\mathcal{Z}(X))$.

By way of illustration, consider the situation in which the set $\mathcal{B}$ comprises the natural numbers lying in some fixed quadratic sequence. Here we take advantage of the fact that the exponential sum $k(\alpha)$ is a complete Weyl sum, and hence provides crucial assistance in the analysis of the left hand side of

(1.2). In order to obtain an effective upper bound for the right hand side of (1.2), on the other hand, one might seek to exploit the underlying arithmetic structure of the set $\mathcal{B}$ via an application of Hölder's inequality in the shape

$$\int_0^1 f_1(\alpha)\dots f_s(\alpha)K(-\alpha)d\alpha \le \Big(\int_0^1 |f_1(\alpha)\dots f_s(\alpha)|^{4/3}d\alpha\Big)^{3/4}\Big(\int_0^1 |K(\alpha)|^4 d\alpha\Big)^{1/4}. \qquad (1.3)$$

Here, a straightforward counting argument shows that

$$\int_0^1 |K(\alpha)|^4 d\alpha \ll X^\varepsilon(\mathrm{card}(\mathcal{Z}(X)))^2,$$

and then by combining (1.2) and (1.3), one obtains the lower bound

$$\mathrm{card}(\mathcal{Z}(X)) \gg X^{-\varepsilon}\Big(\int_0^1 f_1(\alpha)\dots f_s(\alpha)k(-\alpha)d\alpha\Big)^2 \times \Big(\int_0^1 |f_1(\alpha)\dots f_s(\alpha)|^{4/3}d\alpha\Big)^{-3/2}.$$

It is clear how, in principle, one now obtains a lower bound for the cardinality of $\mathcal{Z}(X)$ in this case involving quadratic polynomials, and of course the strength of this bound will be determined by available upper bounds for the mean value

$$\int_0^1 |f_1(\alpha)\dots f_s(\alpha)|^{4/3}d\alpha.$$

Naturally, the approach one takes in estimating the right hand side of (1.2) will vary according to the arithmetic properties of the underlying set $\mathcal{B}$. We direct the reader's attention to our previous work [4, 5, 6] for inspiration concerning possible strategies for addressing such issues.

Throughout, the letters $\varepsilon$ and $\eta$ will denote sufficiently small positive numbers. We take $P$ to be the basic parameter, a large real number depending at most on $\varepsilon$, $\eta$, and any coefficients of implicit polynomials if necessary. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on $\varepsilon$, $\eta$ and implicit polynomials. Summations start at 1 unless indicated otherwise. In an effort to simplify our analysis, we adopt the following convention concerning the parameter $\varepsilon$. Whenever $\varepsilon$ appears in a statement, we assert that for each $\varepsilon > 0$ the statement holds for sufficiently large values of the main parameter. Note that the "value" of $\varepsilon$ may consequently change from statement to statement, and hence also the dependence of implicit constants on $\varepsilon$.

**2. The unfiltered lower bound.** In our first step towards the proof of Theorem 1.1, we establish a lower bound for the number of solutions of a diophantine equation naturally associated with sums of five cubes and a quadratic polynomial. One might regard our lower bound here as an "unfiltered" estimate for $\mathcal{N}_\phi(N)$, since each value of the quadratic polynomial is counted with a weight essentially equal to the number of its representations as the sum of five cubes of positive integers. Later, in §4, we filter out as many of the unweighted values as our methods permit.

We begin by fixing some notation. Let $\phi$ be an integral quadratic polynomial with positive leading coefficient, let $N$ be a large real number and write

$$P = \phi(N)^{1/3}, \quad M = P^{1/8}, \quad R = P^\eta,$$

where $\eta > 0$ is supposed to be sufficiently small. We remark that the first of these relations implies that $P \asymp N^{2/3}$. When $Q$ is a positive number, we define the set of $R$-smooth numbers up to $Q$ by

$$\mathcal{A}(Q,R) = \{n \in [1,Q]\cap\mathbb{Z} \ : \ p \text{ prime}, p|n \Rightarrow p \le R\}.$$

We then define the exponential sums

$$f(\alpha) = \sum_{x\le P} e(\alpha x^3), \quad g(\alpha) = \sum_{y\in\mathcal{A}(P,R)} e(\alpha y^3), \quad k(\alpha) = \sum_{N/2<n\le N} e(\alpha\phi(n)),$$

$$h(\alpha)=\sum_{1\le 2^j\le M^\eta}\ \sum_{\substack{2^jM<p\le 2^{j+1}M\\ p\equiv 2\ (\mathrm{mod}\ 3)}}\ \sum_{z\in\mathcal{A}(P/(2^jM),R)} e(\alpha(pz)^3).$$

Here and elsewhere, we reserve the letter $p$ to denote a prime number. Our arguments are based on an application of the circle method, and it is therefore useful to introduce some notation for Hardy-Littlewood dissections. When $1\le X\le N/2$, define the major arcs $\mathfrak{M}(X)$ as the union of the intervals

$$\mathfrak{M}(q,a;X)=\{\alpha\in[0,1)\ :\ |q\alpha-a|\le XN^{-2}\},$$

with $0\le a\le q\le X$ and $(a,q)=1$. We then define the corresponding set of minor arcs by $\mathfrak{m}(X)=[0,1)\setminus\mathfrak{M}(X)$. We now establish the promised lower bound.

**Lemma 2.1.** *One has*

$$\int_0^1 f(\alpha)g(\alpha)h(\alpha)^3k(-\alpha)d\alpha\gg NP^2. \tag{2.1}$$

*Proof.* We actually obtain, by an application of the Hardy-Littlewood method, an asymptotic formula for the integral on the left hand side of (2.1). In the interest of brevity, we write $\mathfrak{M}=\mathfrak{M}(P^{4/3})$ and $\mathfrak{m}=\mathfrak{m}(P^{4/3})$. We define also a thin set of major arcs $\mathfrak{N}$ by writing $L=(\log P)^{1/100}$, and taking $\mathfrak{N}$ to be the union of the arcs

$$\mathfrak{N}(q,a)=\{\alpha\in[0,1)\ :\ |\alpha-a/q|\le LN^{-2}\},$$

with $0\le a\le q\le L$ and $(a,q)=1$.

The contribution of the minor arcs $\mathfrak{m}$ is easily bounded by means of Weyl's inequality (see, for example, Vaughan [16], Lemma 2.4), together with available mean value estimates for cubic exponential sums. Thus we have

$$\sup_{\alpha\in\mathfrak{m}}|k(\alpha)|\ll NP^{\varepsilon-2/3},$$

and by Hua's lemma (see Lemma 2.5 of Vaughan [16]),

$$\int_0^1|h(\alpha)|^4d\alpha\ll P^{2+\varepsilon},\quad \int_0^1|f(\alpha)|^4d\alpha\ll P^{2+\varepsilon}.$$

Also, by an obvious adjustment of the proof of Lemma 1 of Vaughan [15],

$$\int_0^1|g(\alpha)h(\alpha)^2|^2d\alpha\ll P^{13/4+3\eta}. \tag{2.2}$$

Combining the above estimates through the medium of Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathfrak{m}}|f(\alpha)g(\alpha)h(\alpha)^3k(-\alpha)|d\alpha\le&\Big(\sup_{\alpha\in\mathfrak{m}}|k(\alpha)|\Big)\Big(\int_0^1|h(\alpha)|^4d\alpha\Big)^{1/4}\\ &\times\Big(\int_0^1|f(\alpha)|^4d\alpha\Big)^{1/4}\Big(\int_0^1|g(\alpha)h(\alpha)^2|^2d\alpha\Big)^{1/2}\\ \ll&NP^{\frac{47}{24}+2\eta}.\end{aligned} \tag{2.3}$$

Write

$$S_3(q,a)=\sum_{r=1}^{q}e(ar^3/q),\quad S_\phi(q,a)=\sum_{r=1}^{2q}e(a\phi(r)/q),$$

and

$$v_3(\beta)=\int_0^P e(\beta\gamma^3)d\gamma,\quad v_\phi(\beta)=\int_{N/2}^N e(\beta\phi(\gamma))d\gamma.$$

Define functions $f^*,k^*$ on $\mathfrak{M}$ for $\alpha\in\mathfrak{M}(q,a;P^{4/3})\subseteq\mathfrak{M}$ by taking

$$f^*(\alpha)=q^{-1}S_3(q,a)v_3(\alpha-a/q),\quad k^*(\alpha)=(2q)^{-1}S_\phi(q,a)v_\phi(\alpha-a/q).$$

Then the methods of Chapters 2 and 4 of Vaughan [16] show that

$$\sup_{\alpha\in\mathfrak{M}} |f(\alpha)-f^*(\alpha)| \ll P^{2/3+\varepsilon}, \quad \sup_{\alpha\in\mathfrak{M}} |k(\alpha)-k^*(\alpha)| \ll P^{2/3+\varepsilon}, \tag{2.4}$$

and moreover that when $\alpha\in\mathfrak{M}(q,a;P^{4/3})\subseteq\mathfrak{M}$, one has

$$f^*(\alpha) \ll \frac{P}{(q+P^3|q\alpha-a|)^{1/3}}, \quad k^*(\alpha) \ll \frac{N}{(q+N^2|q\alpha-a|)^{1/2}}. \tag{2.5}$$

Here we note that if the integral polynomial $\phi$ does not have integral coefficients, then necessarily $\phi$ has half-integral coefficients, and it is this observation which motivates our definition of $S_\phi(q,a)$ above.

On making use of (2.4) in the argument used above to treat the minor arcs $\mathfrak{m}$, one readily finds that

$$\int_{\mathfrak{M}} |(k(\alpha)-k^*(\alpha))f(\alpha)g(\alpha)h(\alpha)^3|d\alpha \ll NP^{\frac{47}{24}+2\eta}. \tag{2.6}$$

Next, on noting (2.5), it follows from Lemma 2 of Brüdern [1] that

$$\int_{\mathfrak{M}} |k^*(\alpha)h(\alpha)|^2 d\alpha \ll P^{7/3+\varepsilon},$$

and thus an application of Schwarz's inequality, combined with (2.2) and (2.4), produces the estimate

$$\begin{aligned}\int_{\mathfrak{M}} &|(f(\alpha)-f^*(\alpha))k^*(\alpha)g(\alpha)h(\alpha)^3|d\alpha \\ &\ll P^{2/3+\varepsilon}\Big(\int_{\mathfrak{M}} |k^*(\alpha)h(\alpha)|^2 d\alpha\Big)^{1/2}\Big(\int_0^1 |g(\alpha)h(\alpha)^2|^2 d\alpha\Big)^{1/2} \\ &\ll NP^{\frac{47}{24}+2\eta}.\end{aligned} \tag{2.7}$$

On collecting together (2.3), (2.6) and (2.7), we may conclude thus far that

$$\int_0^1 f(\alpha)g(\alpha)h(\alpha)^3k(-\alpha)d\alpha = \int_{\mathfrak{M}} f^*(\alpha)k^*(-\alpha)g(\alpha)h(\alpha)^3 d\alpha + O(NP^{\frac{47}{24}+2\eta}). \tag{2.8}$$

We now prune back to the thin major arcs $\mathfrak{N}$. From (2.5) we readily deduce that whenever $t>4$ one has

$$\int_{\mathfrak{M}} |k^*(\alpha)|^t d\alpha \ll_t N^{t-2}. \tag{2.9}$$

Also, a standard application of the Hardy-Littlewood method, based on Lemma 4.9 of Vaughan [16], shows that for $t>4$ and $X\geq 1$, one has

$$\int_{\mathfrak{M}\setminus\mathfrak{M}(X)} |f^*(\alpha)|^t d\alpha \ll P^{t-3}X^{\varepsilon-(t-4)/3}. \tag{2.10}$$

Next, on considering the underlying diophantine equation, it follows from Theorem 2 of Vaughan [13] that

$$\int_0^1 |h(\alpha)|^8 d\alpha \ll P^5. \tag{2.11}$$

Finally, as a consequence of Theorem 2 of Brüdern and Wooley [7] it follows that whenever $t>\frac{77}{10}$, one has

$$\int_0^1 |g(\alpha)|^t d\alpha \ll P^{t-3}. \tag{2.12}$$

An application of Hölder's inequality now yields

$$\begin{aligned}\int_{\mathfrak{M}\setminus\mathfrak{N}} &|f^*(\alpha)k^*(-\alpha)g(\alpha)h(\alpha)^3|d\alpha \\ &\ll \Big(\int_{\mathfrak{M}} |k^*(\alpha)|^{\frac{480}{119}} d\alpha\Big)^{\frac{119}{480}}\Big(\int_{\mathfrak{M}\setminus\mathfrak{N}} |f^*(\alpha)|^{\frac{480}{119}} d\alpha\Big)^{\frac{119}{480}} \\ &\quad\times\Big(\int_0^1 |h(\alpha)|^8 d\alpha\Big)^{3/8}\Big(\int_0^1 |g(\alpha)|^{\frac{240}{31}} d\alpha\Big)^{\frac{31}{240}},\end{aligned}$$

so that on noting that $\mathfrak{M}(L) \subseteq \mathfrak{N}$, and collecting together (2.9)–(2.12), we obtain the estimate

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} |f^*(\alpha)k^*(-\alpha)g(\alpha)h(\alpha)^3|d\alpha \ll NP^2(\log P)^{-\tau},$$

for a suitable positive number $\tau$. Consequently, on recalling (2.8), we now infer that

$$\int_0^1 f(\alpha)g(\alpha)h(\alpha)^3k(-\alpha)d\alpha = \int_{\mathfrak{N}} f^*(\alpha)k^*(-\alpha)g(\alpha)h(\alpha)^3d\alpha + O(NP^2(\log P)^{-\tau}). \tag{2.13}$$

The major arcs $\mathfrak{N}$ are sufficiently narrow and sparse that the methods of Vaughan [14] and §4.4 of Vaughan [16] may be successfully applied. It follows from Lemma 8.5 of Wooley [17] (see also Lemma 5.4 of Vaughan [14] for a related conclusion) that there exists a positive number $c$, depending only on $\eta$, such that

$$\sup_{\alpha\in\mathfrak{N}} |g(\alpha) - cf^*(\alpha)| \ll PL^{-10}. \tag{2.14}$$

Suppose next that $\alpha \in \mathfrak{N}(q,a) \subseteq \mathfrak{N}$. Write

$$v(\beta;V) = \int_0^V e(\beta\gamma^3)d\gamma.$$

Then on recalling the definition of $h(\alpha)$, we find from Lemma 8.5 of [17] that there exist positive numbers $c_j$ $(1 \le 2^j \le M^\eta)$, each uniformly bounded away from zero, such that

$$h(\alpha) = \sum_{j,p} \left( c_j q^{-1} S_3(q,ap^3)v(p^3\beta;P/(2^jM)) + O(PL^{-10}/(2^jM)) \right),$$

and here the summations are over $j$ and $p$ with

$$1 \le 2^j \le M^\eta, \quad 2^jM < p \le 2^{j+1}M \quad \text{and} \quad p \equiv 2 \pmod 3.$$

However, the condition $p > 2^jM$ occurring in the summation, together with the implicit hypothesis that $q \le L < M$, ensures that $p \nmid q$. Consequently, for each prime $p$ occurring in the latter summation, it follows via a change of variable in the implicit summation that $S_3(q,ap^3) = S_3(q,a)$. Define the function $h^*(\alpha)$ on $\alpha \in \mathfrak{N}(q,a) \subseteq \mathfrak{N}$ by taking

$$h^*(\alpha) = q^{-1}S_3(q,a)w(\alpha - a/q),$$

where we write

$$w(\beta) = \sum_{1\le 2^j\le M^\eta} c_j \sum_{\substack{2^jM<p\le 2^{j+1}M \\ p\equiv 2 \ (\mathrm{mod}\ 3)}} p^{-1}v(\beta;Pp/(2^jM)). \tag{2.15}$$

Then following an obvious change of variable, we arrive at the estimate

$$\sup_{\alpha\in\mathfrak{N}} |h(\alpha) - h^*(\alpha)| \ll PL^{-10}. \tag{2.16}$$

Collecting together (2.14) and (2.16), and writing

$$T(q,a) = (2q^6)^{-1}S_\phi(q,-a)S_3(q,a)^5 \quad \text{and} \quad u(\beta) = v_\phi(-\beta)v_3(\beta)^2w(\beta)^3,$$

we deduce that when $\alpha \in \mathfrak{N}(q,a) \subseteq \mathfrak{N}$, one has

$$|f^*(\alpha)k^*(-\alpha)g(\alpha)h(\alpha)^3 - cT(q,a)u(\alpha - a/q)| \ll NP^5L^{-10}.$$

Since the measure of $\mathfrak{N}$ is $O(L^3P^{-3})$, it follows that

$$\int_{\mathfrak{N}} f^*(\alpha)k^*(-\alpha)g(\alpha)h(\alpha)^3d\alpha = c\mathfrak{S}(L)J(L) + O(NP^2L^{-1}), \tag{2.17}$$

where we write

$$\mathfrak{S}(L)=\sum_{1\le q\le L}\sum_{\substack{a=1\\(a,q)=1}}^{q}T(q,a)\quad\text{and}\quad J(L)=\int_{-L/N^2}^{L/N^2}u(\beta)d\beta. \tag{2.18}$$

The bounds

$$T(q,a)\ll q^{-13/6}\quad\text{and}\quad u(\beta)\ll NP^5(1+N^2|\beta|)^{-13/6}$$

are essentially immediate, respectively, from Theorem 4.2 of Vaughan [16], and from (2.15) above and Lemma 2.8 of Vaughan [16] via the prime number theorem. Thus a routine argument permits the replacement of the integral in (2.18) by the singular integral

$$J=\int_{-\infty}^{\infty}u(\beta)d\beta,$$

and also allows the completion of the sum in (2.18) to the singular series

$$\mathfrak{S}=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}T(q,a),$$

with acceptable errors which contribute at most $O(NP^2L^{-1/10})$ within (2.17).

Standard endgame technique from the theory of the Hardy-Littlewood method (see §2.6 of Vaughan [16]) shows with ease that $\mathfrak{S}\gg 1$. Meanwhile, whenever $p_i$ is a prime number with $2^{j_i}M<p_i\le 2^{j_i+1}M$ $(i=1,2,3)$, an application of Fourier's integral formula rapidly establishes that

$$\int_{-\infty}^{\infty}v_\phi(-\beta)v_3(\beta)^2\prod_{i=1}^{3}v(\beta;Pp_i/(2^{j_i}M))d\beta\gg NP^2,$$

whence, by the prime number theorem,

$$J\gg NP^2\Biggl(\sum_{1\le 2^j\le M^\eta}c_j\sum_{\substack{2^jM<p\le 2^{j+1}M\\ p\equiv 2\pmod 3}}p^{-1}\Biggr)^3\gg NP^2.$$

We therefore conclude from (2.17) that

$$\int_{\mathfrak{N}}f^*(\alpha)k^*(-\alpha)g(\alpha)h(\alpha)^3d\alpha\gg NP^2,$$

whence the desired conclusion (2.1) follows immediately from (2.13). This completes the proof of the lemma.

**3. An auxiliary mean value estimate.** In this section we derive an upper bound for an auxiliary mean value crucial to the execution of the program outlined in the introduction. Here we again apply the Hardy-Littlewood method, but since we do not seek an asymptotic formula, our argument is in some ways less complex. It is convenient for future use to record the following mean value estimate.

**Lemma 3.1.** *Let $U(X)$ denote the number of solutions of the diophantine equation*

$$x_1^3-x_2^3=y_1^3+y_2^3-y_3^3-y_4^3,$$

*with $1\le x_i\le 2X$ $(i=1,2)$ and $y_j\in\mathcal{A}(X,X^\eta)$ $(1\le j\le 4)$. Then provided that $\eta$ is sufficiently small, one has*

$$U(X)\ll X^{13/4-2\eta}.$$

*Proof.* The conclusion of the lemma follows from Theorem 1.2 of Wooley [18].

We now come to the ingredient crucial to the strength of the lower bound recorded in the statement of Theorem 1.1. It may be worth noting that the integral estimated in the following lemma is expected to be of order $P^{11/3}$.

**Lemma 3.2.** *One has*

$$\int_0^1 |f(\alpha)g(\alpha)h(\alpha)^3|^{4/3}d\alpha \ll P^{11/3+\delta},$$

*where $\delta = 7/136 - \tau$, for a sufficiently small but positive number $\tau$.*

*Proof.* With the notation defined in the preamble to Lemma 2.1, we now write $\mathfrak{P} = \mathfrak{M}(P^{11/8+3\eta})$ and $\mathfrak{p} = [0,1)\backslash\mathfrak{P}$. Then, by applying Theorem 3 of Brüdern and Wooley [8] with $Y = 2^j M$ for $1 \le 2^j \le M^\eta$, it follows from an application of Hölder's inequality that

$$\int_{\mathfrak{p}} |f(\alpha)^2 h(\alpha)^5| d\alpha \ll P^{4-\frac{13}{272}}. \tag{3.1}$$

Moreover, on considering the underlying diophantine equation, it follows from Lemma 3.1 that

$$\int_0^1 |h(\alpha)g(\alpha)^2|^2 d\alpha \ll P^{13/4-\eta}. \tag{3.2}$$

Then on applying Hölder's inequality, we deduce from (3.1) and (3.2) that

$$\begin{aligned}\int_{\mathfrak{p}} |f(\alpha)g(\alpha)h(\alpha)^3|^{4/3}d\alpha &\le \Big(\int_{\mathfrak{p}} |f(\alpha)^2h(\alpha)^5|d\alpha\Big)^{2/3}\Big(\int_0^1 |h(\alpha)g(\alpha)^2|^2 d\alpha\Big)^{1/3}\\ &\ll P^{11/3+\delta}. \end{aligned}\tag{3.3}$$

For the treatment of the major arcs we use the function $f^*(\alpha)$ applied in the proof of Lemma 2.1, extended to $\mathfrak{P}$ in the natural way. By Lemma 4.9 of Vaughan [16] and straightforward estimates, one finds that

$$\int_{\mathfrak{P}} |f^*(\alpha)|^4 d\alpha \ll P^{1+\varepsilon}, \tag{3.4}$$

and from Theorem 4.1 of Vaughan [16] we have

$$\sup_{\alpha\in\mathfrak{P}} |f(\alpha) - f^*(\alpha)| \ll P^{11/16+2\eta}. \tag{3.5}$$

Note first that the bound

$$\int_0^1 |f(\alpha)h(\alpha)^2|^2 d\alpha \ll P^{13/4+3\eta}$$

follows from the proof of Lemma 1 of Vaughan [15], in much the same way as (2.2). Then, applying (3.5) and Hölder's inequality together with the estimate (2.2) and the one just obtained, we derive the bound

$$\begin{aligned}&\int_{\mathfrak{P}} |f(\alpha)-f^*(\alpha)|^{2/3}|f(\alpha)|^{2/3}|g(\alpha)h(\alpha)^3|^{4/3}d\alpha\\ &\qquad \ll (P^{\frac{11}{16}+2\eta})^{2/3}\Big(\int_0^1 |f(\alpha)h(\alpha)^2|^2 d\alpha\Big)^{1/3}\Big(\int_0^1 |g(\alpha)h(\alpha)^2|^2 d\alpha\Big)^{2/3}\\ &\qquad \ll P^{\frac{89}{24}+6\eta}.\end{aligned}\tag{3.6}$$

Consequently, on combining (3.3) and (3.6), we obtain

$$\int_0^1 |f(\alpha)g(\alpha)h(\alpha)^3|^{4/3}d\alpha \ll \int_{\mathfrak{P}} |f(\alpha)f^*(\alpha)|^{2/3}|g(\alpha)h(\alpha)^3|^{4/3}d\alpha + P^{11/3+\delta}. \tag{3.7}$$

Next we apply Hölder's inequality in combination with (2.2), (2.11), (3.4) and (3.5) to obtain

$$\begin{aligned}&\int_{\mathfrak{P}} |f^*(\alpha)|^{2/3}|f(\alpha)-f^*(\alpha)|^{2/3}|g(\alpha)h(\alpha)^3|^{4/3}d\alpha\\ &\qquad \ll (P^{11/16+2\eta})^{2/3}\Big(\int_{\mathfrak{P}} |f^*(\alpha)|^4 d\alpha\Big)^{1/6}\\ &\qquad\quad \times \Big(\int_0^1 |g(\alpha)h(\alpha)^2|^2 d\alpha\Big)^{2/3}\Big(\int_0^1 |h(\alpha)|^8 d\alpha\Big)^{1/6}\\ &\qquad \ll P^{29/8+5\eta}.\end{aligned}$$

Thus, by (3.7) we have

$$\int_0^1 |f(\alpha)g(\alpha)h(\alpha)^3|^{4/3}d\alpha \ll \int_{\mathfrak{P}} |f^*(\alpha)g(\alpha)h(\alpha)^3|^{4/3}d\alpha + P^{11/3+\delta}. \tag{3.8}$$

But by considering the underlying diophantine equation, it follows from Hua's Lemma (see Lemma 2.5 of Vaughan [16]) that

$$\int_0^1 |g(\alpha)h(\alpha)^3|^2 d\alpha \ll P^{5+\varepsilon},$$

and thus an application of Hölder's inequality in combination with (3.4) reveals that

$$\begin{aligned}\int_{\mathfrak{P}} |f^*(\alpha)g(\alpha)h(\alpha)^3|^{4/3}d\alpha &\ll \Big(\int_0^1 |g(\alpha)h(\alpha)^3|^2 d\alpha\Big)^{2/3}\Big(\int_{\mathfrak{P}} |f^*(\alpha)|^4 d\alpha\Big)^{1/3}\\ &\ll P^{11/3+\varepsilon}.\end{aligned}$$

The conclusion of the lemma therefore follows immediately from (3.8).

**4. Filtration: the proof of Theorem 1.1.** It is now a simple matter to establish Theorem 1.1, the insight required becoming transparent presently. Let $\mathcal{Z}(N)$ denote the set of integers $n$ with $N/2 < n \le N$ for which the equation

$$\phi(n) = x^3 + y^3 + (p_1z_1)^3 + (p_2z_2)^3 + (p_3z_3)^3$$

is soluble with $x \le P$, $y \in \mathcal{A}(P,R)$, and for some $j = j(i)$ with $1 \le 2^j \le M^\eta$, the variables $p_i$ and $z_i$ satisfying $2^jM < p_i \le 2^{j+1}M$, $p_i \equiv 2 \pmod 3$ and $z_i \in \mathcal{A}(P/(2^jM),R)$, for $i = 1,2,3$. Then plainly the number of integers $n$ with $N/2 < n \le N$ for which $\phi(n)$ is the sum of five cubes, is at least $\mathrm{card}(\mathcal{Z}(N))$. Moreover, it is apparent from the definition of $\mathcal{Z}(N)$ that necessarily

$$\begin{aligned}\sum_{N/2<n\le N} \int_0^1 & f(\alpha)g(\alpha)h(\alpha)^3 e(-\alpha\phi(n))d\alpha\\ &= \sum_{n\in\mathcal{Z}(N)} \int_0^1 f(\alpha)g(\alpha)h(\alpha)^3 e(-\alpha\phi(n))d\alpha,\end{aligned}$$

whence, on writing

$$K(\alpha) = \sum_{n\in\mathcal{Z}(N)} e(\alpha\phi(n)),$$

we may conclude that

$$\int_0^1 f(\alpha)g(\alpha)h(\alpha)^3 k(-\alpha)d\alpha = \int_0^1 f(\alpha)g(\alpha)h(\alpha)^3 K(-\alpha)d\alpha. \tag{4.1}$$

On applying Hölder's inequality to (4.1), and making use of Lemma 2.1, we obtain

$$NP^2 \ll \Big(\int_0^1 |K(\alpha)|^4 d\alpha\Big)^{1/4}\Big(\int_0^1 |f(\alpha)g(\alpha)h(\alpha)^3|^{4/3}d\alpha\Big)^{3/4}.$$

But by a simple counting argument (see, for example, the proof of the upper bound (3.16) in Brüdern, Kawada and Wooley [4]), one finds that

$$\int_0^1 |K(\alpha)|^4 d\alpha \ll P^\varepsilon \left(\mathrm{card}(\mathcal{Z}(N))\right)^2. \tag{4.2}$$

We therefore conclude from Lemma 3.2 that

$$NP^2 \ll (P^\varepsilon \mathrm{card}(\mathcal{Z}(N)))^{1/2} P^{\frac{11}{4}+\frac{3}{4}\delta},$$

whence

$$\mathrm{card}(\mathcal{Z}(N)) \gg N^2 P^{-\frac{3}{2}-\frac{3}{2}\delta-\varepsilon}.$$

On recalling that $P \asymp N^{2/3}$ and $\delta < \frac{7}{136}$, the proof of Theorem 1.1 is complete.

**5. Three cubes and a square.** Our second illustration of the methods sketched in the introduction concerns values of quadratic polynomials represented by sums of three cubes and a square of positive integers. Let $\phi(t)$ be an integral quadratic polynomial of the type discussed in the preamble to the statement of Theorem 1.2, and let $N$ be sufficiently large in the sense described in the latter. Write $P = N^{2/3}$. Let $k(\alpha)$ be the exponential sum defined in §2, and write also

$$t(\alpha) = \sum_{x \le P^{3/2}} e(\alpha x^2) \quad \text{and} \quad G(\alpha) = \sum_{x \le 2P} \omega(x/P) e(\alpha x^3),$$

where

$$\omega(t) = \exp(-1/(1-(t-1)^2)).$$

Then by orthogonality, on considering the underlying diophantine equation, it follows from work of Hooley [12] and Daniel [9], [10] that

$$\int_0^1 t(\alpha) G(\alpha)^3 k(-\alpha)\, d\alpha \gg N^2. \tag{5.1}$$

Next, let $\mathcal{Z}(N)$ denote the set of integers $n$ with $N/2 < n \le N$ for which the equation

$$\phi(n) = x^2 + y_1^3 + y_2^3 + y_3^3$$

is soluble with $1 \le x \le P^{3/2}$ and $1 \le y_i \le 2P$. Then $\mathrm{card}(\mathcal{Z}(N))$ plainly provides a lower bound for $\mathcal{X}_\phi(N)$. Moreover, on defining the exponential sum $K(\alpha)$ as in §4, we may argue as in the latter section to establish the identity

$$\int_0^1 t(\alpha) G(\alpha)^3 k(-\alpha)\, d\alpha = \int_0^1 t(\alpha) G(\alpha)^3 K(-\alpha)\, d\alpha,$$

whence by (5.1) and Hölder's inequality,

$$N^2 \ll \Big( \int_0^1 |K(\alpha)|^4 d\alpha \Big)^{1/4} \Big( \int_0^1 |t(\alpha) G(\alpha)^3|^{4/3} d\alpha \Big)^{3/4}. \tag{5.2}$$

The first integral on the right hand side of (5.2) may be bounded as in (4.2), and thus we deduce that

$$\mathrm{card}(\mathcal{Z}(N)) \gg N^{4-\varepsilon} \Big( \int_0^1 |t(\alpha) G(\alpha)^3|^{4/3} d\alpha \Big)^{-3/2}. \tag{5.3}$$

We next investigate the mean value on the right hand side of (5.3). With the notation introduced in the preamble to Lemma 2.1, write $\mathfrak{K} = \mathfrak{M}(P^{10/7})$, $\mathfrak{k} = [0,1) \setminus \mathfrak{K}$, and $\mathfrak{M}^*(X) = \mathfrak{M}(X) \setminus \mathfrak{M}(X/2)$. Then by Weyl's inequality, one has

$$\sup_{\alpha \in \mathfrak{k}} |t(\alpha)| \ll P^{11/14+\varepsilon}, \qquad \sup_{\alpha \in \mathfrak{M}^*(X)} |t(\alpha)| \ll P^{3/2+\varepsilon} X^{-1/2}, \tag{5.4}$$

where we suppose that $1 \le X \le P^{3/2}$. Then by Hua's lemma (see Lemma 2.5 of Vaughan [16]), on considering the underlying diophantine equation, we have

$$\int_{\mathfrak{k}} |t(\alpha) G(\alpha)^3|^{4/3} d\alpha \ll P^{22/21+\varepsilon} \int_0^1 |G(\alpha)|^4 d\alpha \ll P^{64/21+\varepsilon}. \tag{5.5}$$

Further, on noting that Theorem 2 of Brüdern [2] delivers, for $1 \le X \le P^{10/7}$, the upper bound

$$\int_{\mathfrak{M}^*(X)} |G(\alpha)|^4 d\alpha \ll P^\varepsilon (X^{7/2} P^{-3} + X^2 P^{-1} + P),$$

we deduce from (5.4) that for $1 \le X \le P^{10/7}$, one has

$$\begin{aligned} \int_{\mathfrak{M}^*(X)} |t(\alpha) G(\alpha)^3|^{4/3} d\alpha &\ll P^{2+\varepsilon} X^{-2/3} (X^{7/2} P^{-3} + X^2 P^{-1} + P) \\ &\ll P^{64/21+\varepsilon}. \end{aligned}$$

Consequently, one obtains by a dyadic dissection of $\mathfrak{K}$ the estimate

$$\int_{\mathfrak{K}} |t(\alpha) G(\alpha)^3|^{4/3} d\alpha \ll P^{64/21+\varepsilon},$$

whence by (5.5),

$$\int_0^1 |t(\alpha) G(\alpha)^3|^{4/3} d\alpha \ll P^{64/21+\varepsilon}.$$

Substituting the latter bound into (5.3), we deduce that $\mathrm{card}(\mathcal{Z}(N)) \gg N^{20/21-\varepsilon}$, and this establishes the conclusion of Theorem 1.2.

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